SOME REVERSES OF THE CONTINUOUS TRIANGLE INEQUALITY FOR BOCHNER INTEGRAL OF VECTOR-VALUED FUNCTIONS IN COMPLEX HILBERT SPACES

SEVER S. DRAGOMIR

Abstract. Some reverses of the continuous triangle inequality for Bochner integral of vector-valued functions in complex Hilbert spaces are given. Applications for complex-valued functions are provided as well.

1. Introduction

Let $f:[a,b]\to\mathbb{K}$, $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality is the continuous version of the triangle inequality

$$\left|\int_a^b f(x)\,dx\right|\le\int_a^b |f(x)|\,dx, \tag{1.1}$$

and plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [5, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949, [3]:

$$\cos\theta\int_a^b |f(x)|\,dx\le\left|\int_a^b f(x)\,dx\right| \tag{1.2}$$

provided

$$-\theta\le\arg[f(x)]\le\theta,\quad x\in[a,b]$$

for given $\theta\in\left(0,\frac{\pi}{2}\right)$.

In [2], the author has extended the above result for Bochner integrals of vector-valued functions in real or complex Hilbert spaces.

If $(H;\langle\cdot,\cdot\rangle)$ is a Hilbert space over $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $f\in L([a,b];H)$, this means that $f:[a,b]\to H$ is Bochner measurable on $[a,b]$ and $\int_a^b\|f(t)\|\,dt$ is finite, and there exists a constant $K\ge 1$ and a vector $e\in H$, $\|e\|=1$ such that

$$\|f(t)\|\le K\operatorname{Re}\langle f(t),e\rangle\quad\text{for a.e.}\quad t\in[a,b], \tag{1.3}$$

then we have the inequality

$$\int_a^b\|f(t)\|\,dt\le K\left\|\int_a^b f(t)\,dt\right\|. \tag{1.4}$$

Date: April 16, 2004.

2000 *Mathematics Subject Classification.* Primary 46C05; Secondary 26D15.

Key words and phrases. Triangle inequality, Diaz-Metcalf inequality, Integral inequalities, Complex Hilbert spaces.

This provides a reverse inequality for the well know result for Bochner integrals and vector-valued functions:

$$\left\| \int_a^b f(t)\, dt \right\| \leq \int_a^b \| f(t) \| \, dt \tag{1.5}$$

for any $f \in L([a,b]; H)$.

Note that the case of equality holds in (1.4) (see [2]) if and only if

$$\int_a^b f(t)\, dt = \frac{1}{K} \left(\int_a^b \| f(t) \| \, dt \right) e. \tag{1.6}$$

For some particular cases of interest, see [2].

The main aim of the present paper is to point out some newer inequalities for complex Hilbert spaces under various conditions for both $\operatorname{Re}\langle f(t), e\rangle$ and $\operatorname{Im}\langle f(t), e\rangle$ $(e \in H, \ \|e\| = 1)$ and in this way improve some earlier results from [2] that have been stated for real or complex Hilbert spaces. Applications for complex-valued functions are also provided.

2. The Case of a Unit Vector

The following result holds.

Theorem 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. If $f \in L([a,b]; H)$ is such that there exists $k_1, k_2 \geq 0$ with*

$$k_1 \| f(t) \| \leq \operatorname{Re}\langle f(t), e\rangle, \quad k_2 \| f(t) \| \leq \operatorname{Im}\langle f(t), e\rangle \tag{2.1}$$

for a.e. $t \in [a,b]$, where $e \in H$, $\|e\| = 1$, is given, then

$$\sqrt{k_1^2 + k_2^2} \int_a^b \| f(t) \| \, dt \leq \left\| \int_a^b f(t)\, dt \right\|. \tag{2.2}$$

The case of equality holds in (2.2) if and only if

$$\int_a^b f(t)\, dt = (k_1 + i k_2) \left(\int_a^b \| f(t) \| \, dt \right) e. \tag{2.3}$$

Proof. Using the Schwarz inequality $\|u\| \|v\| \geq |\langle u, v\rangle|$, $u, v \in H$; in the complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$, we have

$$\begin{aligned}
\left\| \int_a^b f(t)\, dt \right\|^2 &= \left\| \int_a^b f(t)\, dt \right\|^2 \|e\|^2 \\
&\geq \left| \left\langle \int_a^b f(t)\, dt, e \right\rangle \right|^2 = \left| \int_a^b \langle f(t), e\rangle \, dt \right|^2 \\
&= \left| \int_a^b \operatorname{Re}\langle f(t), e\rangle \, dt + i \left(\int_a^b \operatorname{Im}\langle f(t), e\rangle \, dt \right) \right|^2 \\
&= \left(\int_a^b \operatorname{Re}\langle f(t), e\rangle \, dt \right)^2 + \left(\int_a^b \operatorname{Im}\langle f(t), e\rangle \, dt \right)^2 .
\end{aligned} \tag{2.4}$$

Now, on integrating (2.1), we deduce

$$k_1 \int_a^b \|f(t)\| \, dt \leq \int_a^b \operatorname{Re} \langle f(t), e \rangle \, dt, \quad k_2 \int_a^b \|f(t)\| \, dt \leq \int_a^b \operatorname{Im} \langle f(t), e \rangle \, dt \tag{2.5}$$

implying

$$\left(\int_a^b \operatorname{Re} \langle f(t), e \rangle \, dt \right)^2 \geq k_1^2 \left(\int_a^b \|f(t)\| \, dt \right)^2 \tag{2.6}$$

and

$$\left(\int_a^b \operatorname{Im} \langle f(t), e \rangle \, dt \right)^2 \geq k_2^2 \left(\int_a^b \|f(t)\| \, dt \right)^2. \tag{2.7}$$

If we add (2.6) and (2.7) and use (2.4), we deduce the desired inequality (2.2).

Further, if (2.3) holds, then obviously

$$\begin{aligned} \left\| \int_a^b f(t) \, dt \right\| &= |k_1 + ik_2| \left(\int_a^b \|f(t)\| \, dt \right) \|e\| \\ &= \sqrt{k_1^2 + k_2^2} \int_a^b \|f(t)\| \, dt, \end{aligned}$$

and the equality case holds in (2.2).

Before we prove the reverse implication, let us observe that, for $x \in H$ and $e \in H$, $\|e\| = 1$, the following identity is valid

$$\|x - \langle x, e \rangle e\|^2 = \|x\|^2 - |\langle x, e \rangle|^2,$$

therefore $\|x\| = |\langle x, e \rangle|$ if and only if $x = \langle x, e \rangle e$.

If we assume that equality holds in (1.2), then the case of equality must hold in all the inequalities required in the argument used to prove the inequality (2.2). Therefore, we must have

$$\left\| \int_a^b f(t) \, dt \right\| = \left| \left\langle \int_a^b f(t) \, dt, e \right\rangle \right| \tag{2.8}$$

and

$$k_1 \|f(t)\| = \operatorname{Re} \langle f(t), e \rangle, \quad k_2 \|f(t)\| = \operatorname{Im} \langle f(t), e \rangle \tag{2.9}$$

for a.e. $t \in [a, b]$.

From (2.8) we deduce

$$\int_a^b f(t) \, dt = \left\langle \int_a^b f(t) \, dt, e \right\rangle e, \tag{2.10}$$

and from (2.9), by multiplying the second equality with i,the imaginary unit, and integrating both equations on $[a, b]$, we deduce

$$(k_1 + ik_2) \int_a^b \|f(t)\| \, dt = \left\langle \int_a^b f(t) \, dt, e \right\rangle. \tag{2.11}$$

Finally, by (2.10) and (2.11), we deduce the desired equality (2.3). ∎

The following corollary is of interest.

Corollary 1. *Let e be a unit vector in the complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$ and $\eta_1,\eta_2\in(0,1)$. If $f\in L([a,b];H)$ is such that*

$$\|f(t)-e\|\leq\eta_1,\quad \|f(t)-ie\|\leq\eta_2 \ \text{ for a.e. } \ t\in[a,b], \tag{2.12}$$

then we have the inequality

$$\sqrt{2-\eta_1^2-\eta_2^2}\int_a^b\|f(t)\|\,dt\leq\left\|\int_a^b f(t)\,dt\right\|. \tag{2.13}$$

The case of equality holds in (2.13) if and only if

$$\int_a^b f(t)\,dt=\left(\sqrt{1-\eta_1^2}+i\sqrt{1-\eta_2^2}\right)\left(\int_a^b\|f(t)\|\,dt\right)e. \tag{2.14}$$

Proof. From the first inequality in (2.12) we deduce, by taking the square, that

$$\|f(t)\|^2+1-\eta_1^2\leq 2\operatorname{Re}\langle f(t),e\rangle,$$

implying

$$\frac{\|f(t)\|^2}{\sqrt{1-\eta_1^2}}+\sqrt{1-\eta_1^2}\leq\frac{2\operatorname{Re}\langle f(t),e\rangle}{\sqrt{1-\eta_1^2}} \tag{2.15}$$

for a.e. $t\in[a,b]$.

Since, obviously

$$2\|f(t)\|\leq\frac{\|f(t)\|^2}{\sqrt{1-\eta_1^2}}+\sqrt{1-\eta_1^2}, \tag{2.16}$$

hence, by (2.15) and (2.16) we get

$$0\leq\sqrt{1-\eta_1^2}\,\|f(t)\|\leq\operatorname{Re}\langle f(t),e\rangle \tag{2.17}$$

for a.e. $t\in[a,b]$.

From the second inequality in (2.12) we deduce

$$0\leq\sqrt{1-\eta_2^2}\,\|f(t)\|\leq\operatorname{Re}\langle f(t),ie\rangle$$

for a.e. $t\in[a,b]$. Since

$$\operatorname{Re}\langle f(t),ie\rangle=\operatorname{Im}\langle f(t),e\rangle$$

hence

$$0\leq\sqrt{1-\eta_2^2}\,\|f(t)\|\leq\operatorname{Im}\langle f(t),e\rangle \tag{2.18}$$

for a.e. $t\in[a,b]$.

Now, observe from (2.17) and (2.18), that the condition (2.1) of Theorem 1 is satisfied for $k_1=\sqrt{1-\eta_1^2}$, $k_2=\sqrt{1-\eta_2^2}\in(0,1)$, and thus the corollary is proved. ∎

The following corollary may be stated as well.

Corollary 2. *Let e be a unit vector in the complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$ and $M_1\geq m_1>0$, $M_2\geq m_2>0$. If $f\in L([a,b];H)$ is such that either*

$$\operatorname{Re}\langle M_1e-f(t),f(t)-m_1e\rangle\geq 0,\qquad \operatorname{Re}\langle M_2ie-f(t),f(t)-m_2ie\rangle\geq 0 \tag{2.19}$$

or, equivalently,

$$\left\| f(t) - \frac{M_1 + m_1}{2} e \right\| \leq \frac{1}{2}(M_1 - m_1), \tag{2.20}$$
$$\left\| f(t) - \frac{M_2 + m_2}{2} ie \right\| \leq \frac{1}{2}(M_2 - m_2),$$

for each a.e. $t \in [a, b]$, *then we have the inequality*

$$2\left[\frac{m_1 M_1}{(M_1 + m_1)^2} + \frac{m_2 M_2}{(M_2 + m_2)^2}\right]^{1/2} \int_a^b \|f(t)\| \, dt \leq \left\| \int_a^b f(t) \, dt \right\|. \tag{2.21}$$

The equality holds in (2.21) if and only if

$$\int_a^b f(t) \, dt = 2\left(\frac{\sqrt{m_1 M_1}}{M_1 + m_1} + i\frac{\sqrt{m_2 M_2}}{M_2 + m_2}\right)\left(\int_a^b \|f(t)\| \, dt\right) e. \tag{2.22}$$

Proof. Firstly, remark that, for $x, z, Z \in H$, the following statements are equivalent.

(i) $\operatorname{Re}\langle Z - x, x - z\rangle \geq 0$
and
(ii) $\left\| x - \frac{Z+z}{2} \right\| \leq \frac{1}{2}\|Z - z\|$.

Using this fact, we may simply realise that (2.19) and (2.20) are equivalent.

Now, from the first inequality in (2.19), we get

$$\|f(t)\|^2 + m_1 M_1 \leq (M_1 + m_1) \operatorname{Re}\langle f(t), e\rangle$$

implying

$$\frac{\|f(t)\|^2}{\sqrt{m_1 M_1}} + \sqrt{m_1 M_1} \leq \frac{M_1 + m_1}{\sqrt{m_1 M_1}} \operatorname{Re}\langle f(t), e\rangle \tag{2.23}$$

for a.e. $t \in [a, b]$.

Since, obviously,

$$2\|f(t)\| \leq \frac{\|f(t)\|^2}{\sqrt{m_1 M_1}} + \sqrt{m_1 M_1}, \tag{2.24}$$

hence, by (2.23) and (2.24)

$$0 \leq \frac{2\sqrt{m_1 M_1}}{M_1 + m_1} \|f(t)\| \leq \operatorname{Re}\langle f(t), e\rangle \tag{2.25}$$

for a.e. $t \in [a, b]$.

Using the same argument as in the proof of Corollary 1, we deduce the desired inequality. We omit the details. ∎

3. The Case of Orthonormal Vectors

In the early paper [2], we pointed out the following reverse of the continuous triangle inequality for real or complex Hilbert spaces $(H; \langle\cdot,\cdot\rangle)$.

Theorem 2. *Let* $\{e_1, \dots, e_n\}$ *be a family of orthonormal vectors in* H, $k_i \geq 0$, $i \in \{1, \dots, n\}$ *and* $f \in L([a, b]; H)$ *such that*

$$k_i \|f(t)\| \leq \operatorname{Re}\langle f(t), e_i\rangle \tag{3.1}$$

for each $i \in \{1, \ldots, n\}$ *and for a.e.* $t \in [a, b]$. *Then*

$$\left(\sum_{i=1}^{n} k_i^2\right)^{\frac{1}{2}} \int_a^b \|f(t)\| \, dt \le \left\| \int_a^b f(t) \, dt \right\|, \tag{3.2}$$

where the case of equality holds if and only if

$$\int_a^b f(t) \, dt = \left(\int_a^b \|f(t)\| \, dt\right) \sum_{i=1}^{n} k_i e_i. \tag{3.3}$$

In what follows, we improve this result for the case of complex Hilbert spaces. The following result holds.

Theorem 3. *Let* $\{e_1, \ldots, e_n\}$ *be a family of orthonormal vectors in the complex Hilbert space* $(H; \langle \cdot, \cdot \rangle)$. *If* $k_j, h_j \ge 0$, $j \in \{1, \ldots, n\}$ *and* $f \in L([a, b]; H)$ *are such that*

$$k_j \|f(t)\| \le \operatorname{Re} \langle f(t), e_j \rangle, \quad h_j \|f(t)\| \le \operatorname{Im} \langle f(t), e_j \rangle \tag{3.4}$$

for each $j \in \{1, \ldots, n\}$ *and a.e.* $t \in [a, b]$, *then*

$$\left[\sum_{j=1}^{n} \left(k_j^2 + h_j^2\right)\right]^{\frac{1}{2}} \int_a^b \|f(t)\| \, dt \le \left\| \int_a^b f(t) \, dt \right\|. \tag{3.5}$$

The case of equality holds in (3.5) if and only if

$$\int_a^b f(t) \, dt = \left(\int_a^b \|f(t)\| \, dt\right) \sum_{j=1}^{n} (k_j + i h_j) e_j. \tag{3.6}$$

Proof. Before we prove the theorem, let us recall that, if $x \in H$ and $e_1, \ldots, e_n$ are orthonormal vectors, then the following identity holds true:

$$\left\| x - \sum_{j=1}^{n} \langle x, e_j \rangle e_j \right\|^2 = \|x\|^2 - \sum_{j=1}^{n} |\langle x, e_j \rangle|^2. \tag{3.7}$$

As a consequence of this identity, we have the *Bessel inequality*

$$\sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \le \|x\|^2, x \in H. \tag{3.8}$$

in which the case of equality holds if and only if

$$x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j. \tag{3.9}$$

Now, applying Bessel's inequality for $x = \int_a^b f(t)\,dt$, we have successively

$$\left\| \int_a^b f(t)\,dt \right\|^2 \geq \sum_{j=1}^n \left| \left\langle \int_a^b f(t)\,dt, e_j \right\rangle \right|^2 = \sum_{j=1}^n \left| \int_a^b \langle f(t), e_j \rangle\,dt \right|^2 \tag{3.10}$$

$$= \sum_{j=1}^n \left| \int_a^b \operatorname{Re} \langle f(t), e_j \rangle\,dt + i \left(\int_a^b \operatorname{Im} \langle f(t), e_j \rangle\,dt \right) \right|^2$$

$$= \sum_{j=1}^n \left[\left(\int_a^b \operatorname{Re} \langle f(t), e_j \rangle\,dt \right)^2 + \left(\int_a^b \operatorname{Im} \langle f(t), e_j \rangle\,dt \right)^2 \right].$$

Integrating (3.4) on $[a, b]$, we get

$$\int_a^b \operatorname{Re} \langle f(t), e_j \rangle\,dt \geq k_j \int_a^b \|f(t)\|\,dt \tag{3.11}$$

and

$$\int_a^b \operatorname{Im} \langle f(t), e_j \rangle\,dt \geq h_j \int_a^b \|f(t)\|\,dt \tag{3.12}$$

for each $j \in \{1, \dots, n\}$.

Squaring and adding the above two inequalities (3.11) and (3.12), we deduce

$$\sum_{j=1}^n \left[\left(\int_a^b \operatorname{Re} \langle f(t), e_j \rangle\,dt \right)^2 + \left(\int_a^b \operatorname{Im} \langle f(t), e_j \rangle\,dt \right)^2 \right]$$

$$\geq \sum_{j=1}^n \left(k_j^2 + h_j^2\right) \left(\int_a^b \|f(t)\|\,dt \right)^2,$$

which combined with (3.10) will produce the desired inequality (3.5).

Now, if (3.6) holds true, then

$$\left\| \int_a^b f(t)\,dt \right\| = \left(\int_a^b \|f(t)\|\,dt \right) \left\| \sum_{j=1}^n (k_j + i h_j)\,e_j \right\|$$

$$= \left(\int_a^b \|f(t)\|\,dt \right) \left(\left\| \sum_{j=1}^n (k_j + i h_j)\,e_j \right\|^2 \right)^{\frac{1}{2}}$$

$$= \left(\int_a^b \|f(t)\|\,dt \right) \left[\sum_{j=1}^n \left(k_j^2 + h_j^2\right) \right]^{\frac{1}{2}},$$

and the case of equality holds in (3.5).

Conversely, if the equality holds in (3.5), then it must hold in all the inequalities used to prove (3.5) and therefore we must have

$$\left\| \int_a^b f(t)\,dt \right\|^2 = \sum_{j=1}^n \left| \left\langle \int_a^b f(t)\,dt, e_j \right\rangle \right|^2 \tag{3.13}$$

and

$$k_j \|f(t)\| = \operatorname{Re}\langle f(t), e_j\rangle \quad \text{and} \quad h_j \|f(t)\| = \operatorname{Re}\langle f(t), e_j\rangle \tag{3.14}$$

for each $j \in \{1, \dots, n\}$ and a.e. $t \in [a, b]$.

From (3.13), on using the identity (3.9), we deduce that

$$\int_a^b f(t)\, dt = \sum_{j=1}^{n} \left\langle \int_a^b f(t)\, dt, e_j \right\rangle e_j. \tag{3.15}$$

Now, multiplying the second equality in (3.14) with the imaginary unit i, integrating both inequalities on $[a, b]$ and summing them up, we get

$$(k_j + i h_j) \int_a^b \|f(t)\|\, dt = \left\langle \int_a^b f(t)\, dt, e_j \right\rangle \tag{3.16}$$

for each $j \in \{1, \dots, n\}$.

Finally, utilising (3.15) and (3.16), we deduce (3.6) and the theorem is proved. ∎

The following corollaries are of interest.

Corollary 3. *Let $e_1, \dots, e_m$ be orthonormal vectors in the complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\rho_k, \eta_k \in (0, 1)$, $k \in \{1, \dots, n\}$. If $f \in L([a, b]; H)$ is such that*

$$\|f(t) - e_k\| \le \rho_k, \qquad \|f(t) - i e_k\| \le \eta_k$$

for each $k \in \{1, \dots, n\}$ and for a.e. $t \in [a, b]$, then we have the inequality

$$\left[\sum_{k=1}^{n} \left(2 - \rho_k^2 - \eta_k^2\right)\right]^{\frac{1}{2}} \int_a^b \|f(t)\|\, dt \le \left\| \int_a^b f(t)\, dt \right\|. \tag{3.17}$$

The case of equality holds in (3.17) if and only if

$$\int_a^b f(t)\, dt = \left(\int_a^b \|f(t)\|\, dt\right) \sum_{k=1}^{n} \left(\sqrt{1 - \rho_k^2} + i\sqrt{1 - \eta_k^2}\right) e_k. \tag{3.18}$$

The proof follows by Theorem 3 and is similar to the one from Corollary 1. We omit the details.

Corollary 4. *Let $e_1, \dots, e_m$ be as in Corollary 3 and $M_k \ge m_k > 0$, $N_k \ge n_k > 0$, $k \in \{1, \dots, n\}$. If $f \in L([a, b]; H)$ is such that either*

$$\operatorname{Re}\langle M_k e_k - f(t), f(t) - m_k e_k\rangle \ge 0, \quad \operatorname{Re}\langle N_k i e_k - f(t), f(t) - n_k i e_k\rangle \ge 0$$

or, equivalently,

$$\left\| f(t) - \frac{M_k + m_k}{2} e_k \right\| \le \frac{1}{2}(M_k - m_k),$$
$$\left\| f(t) - \frac{N_k + n_k}{2} i e_k \right\| \le \frac{1}{2}(N_k - n_k)$$

for each $k \in \{1, \dots, n\}$ and a.e. $t \in [a, b]$, then we have the inequality

$$2\left\{\sum_{k=1}^{m} \left[\frac{m_k M_k}{(M_k + m_k)^2} + i \frac{n_k N_k}{(N_k + n_k)^2}\right]\right\}^{\frac{1}{2}} \int_a^b \|f(t)\|\, dt \le \left\| \int_a^b f(t)\, dt \right\|. \tag{3.19}$$

The case of equality holds in (3.19) if and only if

$$\int_a^b f(t)\,dt = 2\left(\int_a^b \|f(t)\|\,dt\right)\sum_{k=1}^n \left(\frac{\sqrt{m_k M_k}}{M_k+m_k} + i\frac{\sqrt{n_k N_k}}{N_k+n_k}\right) e_k. \tag{3.20}$$

The proof employs Theorem 3 and is similar to the one in Corollary 2. We omit the details.

4. Applications for Complex-Valued Functions

The following reverse of the generalised triangle inequality for complex-valued functions that improves Karamata's result (1.2) holds.

Proposition 1. *Let $f \in L([a,b];\mathbb{C})$ with the property that*

$$0 \le \varphi_1 \le \arg f(t) \le \varphi_2 < \frac{\pi}{2} \tag{4.1}$$

for a.e. $t \in [a,b]$. Then we have the inequality

$$\sqrt{\sin^2\varphi_1 + \cos^2\varphi_2}\int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right|. \tag{4.2}$$

The equality holds in (4.2) if and only if

$$\int_a^b f(t)\,dt = (\cos\varphi_2 + i\sin\varphi_1)\int_a^b |f(t)|\,dt. \tag{4.3}$$

Proof. Let $f(t) = \operatorname{Re} f(t) + i\operatorname{Im} f(t)$. We may assume that $\operatorname{Re} f(t) \ge 0$, $\operatorname{Im} f(t) > 0$, for a.e. $t \in [a,b]$, since, by (4.1), $\frac{\operatorname{Im} f(t)}{\operatorname{Re} f(t)} = \tan[\arg f(t)] \in \left[0, \frac{\pi}{2}\right)$, for a.e. $t \in [a,b]$. By (4.1), we obviously have

$$0 \le \tan^2\varphi_1 \le \left[\frac{\operatorname{Im} f(t)}{\operatorname{Re} f(t)}\right]^2 \le \tan^2\varphi_2, \qquad \text{for a.e. } t \in [a,b],$$

from where we get

$$\frac{[\operatorname{Im} f(t)]^2 + [\operatorname{Re} f(t)]^2}{[\operatorname{Re} f(t)]^2} \le \frac{1}{\cos^2\varphi_2},$$

for a.e. $t \in [a,b]$, and

$$\frac{[\operatorname{Im} f(t)]^2 + [\operatorname{Re} f(t)]^2}{[\operatorname{Im} f(t)]^2} \le \frac{1+\tan^2\varphi_1}{\tan^2\varphi_1} = \frac{1}{\sin\varphi_1},$$

for a.e. $t \in [a,b]$, giving the simpler inequalities

$$|f(t)|\cos\varphi_2 \le \operatorname{Re}(f(t)), \quad |f(t)|\sin\varphi_1 \le \operatorname{Im}(f(t))$$

for a.e. $t \in [a,b]$.

Now, applying Theorem 1 for the complex Hilbert space $\mathbb{C}$ endowed with the inner product $\langle z, w\rangle = z \cdot \bar{w}$ for $k_1 = \cos\varphi_2$, $k_2 = \sin\varphi_1$ and $e = 1$, we deduce the desired inequality (4.2). The case of equality is also obvious and we omit the details. ∎

Another result that has an obvious geometrical interpretation is the following one.

Proposition 2. *Let $e \in \mathbb{C}$ with $|e| = 1$ and $\rho_1, \rho_2 \in (0,1)$. If $f(t) \in L([a,b];\mathbb{C})$ such that*

$$|f(t) - e| \leq \rho_1, \quad |f(t) - ie| \leq \rho_2 \qquad \text{for a.e. } t \in [a,b], \tag{4.4}$$

then we have the inequality

$$\sqrt{2 - \rho_1^2 - \rho_2^2} \int_a^b |f(t)|\, dt \leq \left| \int_a^b f(t)\, dt \right|, \tag{4.5}$$

with equality if and only if

$$\int_a^b f(t)\, dt = \left(\sqrt{1-\rho_1^2} + i\sqrt{1-\rho_2^2} \right) \int_a^b |f(t)|\, dt \cdot e. \tag{4.6}$$

The proof is obvious by Corollary 1 applied for $H = \mathbb{C}$ and we omit the details.

Remark 1. *If we choose $e = 1$, and for $\rho_1, \rho_2 \in (0,1)$ we define*

$$\bar{D}(1, \rho_1) := \{ z \in \mathbb{C} |\, |z-1| \leq \rho_1 \}, \quad \bar{D}(i, \rho_2) := \{ z \in \mathbb{C} |\, |z-i| \leq \rho_2 \},$$

then obviously the intersection domain

$$S_{\rho_1,\rho_2} := \bar{D}(1, \rho_1) \cap \bar{D}(i, \rho_2)$$

is nonempty if and only if $\rho_1 + \rho_2 > \sqrt{2}$.

If $f(t) \in S_{\rho_1,\rho_2}$ for a.e. $t \in [a,b]$, then (4.5) holds true. The equality holds in (4.5) if and only if

$$\int_a^b f(t)\, dt = \left(\sqrt{1-\rho_1^2} + i\sqrt{1-\rho_2^2} \right) \int_a^b |f(t)|\, dt.$$

References

[1] J.B. DIAZ and F.T. METCALF, A complementary triangle inequality in Hilbert and Banach spaces, *Proceedings Amer. Math. Soc.,* **17**(1) (1966), 88-97.

[2] S.S. DRAGOMIR, Reverses of the continuous triangle inequality for Bochner integrals of vector-valued functions in Hilbert spaces, *RGMIA Res. Rep. Coll.,* **7** (2004), to appear.

[3] J. KARAMATA, *Teorija i Praksa Stieltjesova Integrala* (Serbo-Coratian) (Stieltjes Integral, Theory and Practice), SANU, Posebna izdanja, 154, Beograd, 1949.

[4] M. MARDEN, The Geometry of the Zeros of a Polynomial in a Complex Variable, *Amer. Math. Soc. Math. Surveys*, **3**, New York, 1949.

[5] D.S. MITRINOVIĆ, J.E. PEČARIĆ and A.M. FINK, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[6] M. PETROVICH, Module d'une somme, *L' Ensignement Mathématique,* **19** (1917), 53-56.

[7] H.S. WILF, Some applications of the inequality of arithmetic and geometric means to polynomial equations, *Proceedings Amer. Math. Soc.,* **14** (1963), 263-265.

School of Computer Science and Mathematics, Victoria University of Technology, PO Box 14428, MCMC 8001, Victoria, Australia.

E-mail address: `sever@csm.vu.edu.au`

URL: `http://rgmia.vu.edu.au/SSDragomirWeb.html`